Chapter 16 – Revision of chapters 12–15

Solutions to 16A Short-answer questions

1 a $60° = 60 \times \frac{\pi}{180}$ radians $= \frac{\pi}{3}$ radians

b $270° = \frac{3\pi}{2}$ radians

c $140° = 140 \times \frac{\pi}{180}$ radians $= \frac{7\pi}{9}$ radians

2 a $\sin\left(-\frac{\pi}{2}\right) = -1$

b $\cos\left(\frac{3\pi}{2}\right) = 0$

c $\tan(3\pi) = 0$

d $\tan\left(-\frac{\pi}{2}\right)$ undefined

3 a $\tan\theta = \frac{2}{5}$

$\theta \approx 21.8°$

b $x = 4\cos 40° \approx 3.06$

c $\frac{6}{x} = \sin 37°$

$x = \frac{6}{\sin 37°} \approx 9.97$

4 a $\sin(2\pi - \theta) = -\sin\theta = -0.3$

b $\cos(-\theta) = \cos\theta = -0.5$

c $\tan(\pi + \theta) = \tan\theta = 1.6$

d $\sin(\pi + \theta) = -\sin\theta = -0.6$

e $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta = 0.1$

f $\cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ (since $0 < \theta < \frac{\pi}{2}$)

5 a $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

c $\tan\left(\frac{-\pi}{4}\right) = -1$

d $\sin\left(\frac{-7\pi}{6}\right) = \frac{1}{2}$

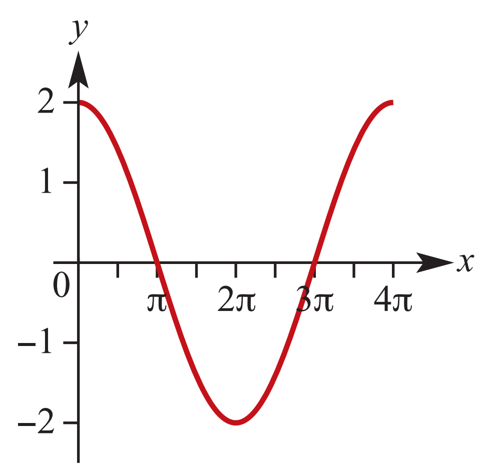
e $\cos\left(\frac{-7\pi}{4}\right) = \frac{1}{\sqrt{2}}$

f $\tan\left(\frac{5\pi}{3}\right) == -\sqrt{3}$

6 $f(x) = 2\cos\left(\frac{x}{2}\right)$.

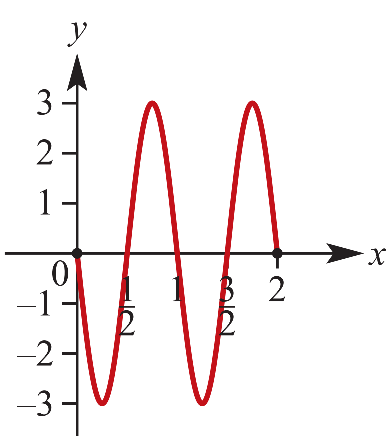
a Period $= 4\pi$; Amplitude $= 2$

b



c Dilation of factor 2 from the x-axis and dilation of factor 2 from the y-axis

7



8 **a** $\cos\theta = -\dfrac{\sqrt{3}}{2}$

$$\theta = -\frac{5\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6},$$

b $\sqrt{2}\sin\theta = 1$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

c

$$\sin(2\theta) = -\frac{1}{2}$$

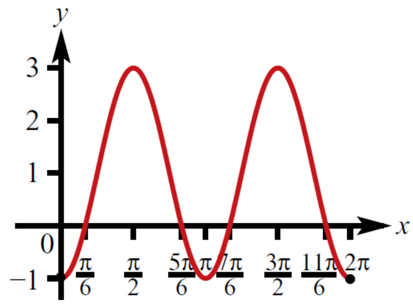
$$2\theta = -\frac{17\pi}{6}, -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$\theta = -\frac{17\pi}{12}, -\frac{13\pi}{12}, -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

d $\tan\theta = -\sqrt{3}$

$$\theta = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

9 This is the graph of $y = \sin(x)$ with a dilation by a factor of 2 from the x-axis, a dilation by a factor of $\frac{1}{2}$ from the y-axis, translation of $\frac{\pi}{4}$ units right and 1 unit up.



To find the x-axis intercepts, solve $2\sin 2\left(x - \frac{\pi}{4}\right) + 1 = 0.$

$$2\sin 2\left(x - \frac{\pi}{4}\right) + 1 = 0$$

$$\sin 2\left(x - \frac{\pi}{4}\right) = -\frac{1}{2}$$

$$2\left(x - \frac{\pi}{4}\right) = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \ldots$$

$$x - \frac{\pi}{4} = -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \ldots$$

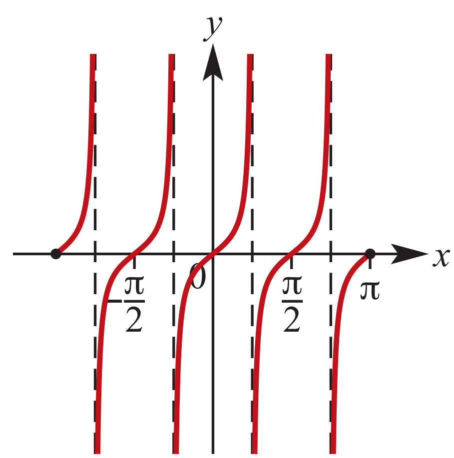
$$x - \frac{3\pi}{12} = -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \ldots$$

$$x = \frac{2\pi}{12}, \frac{10\pi}{12}, \frac{14\pi}{12}, \frac{22\pi}{12}, \frac{26\pi}{12}, \ldots$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Note that $\frac{26\pi}{12}$ is outside the domain of $x \in [0, 2\pi]$.

10



11 Expand $\sin(A-B)$ using the formula:
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

As the values of $\sin A$ and $\sin B$ are already known, only $\cos B$ and $\cos A$ need to be found.

To find $\cos A$:

$$\begin{aligned}\cos A &= \pm\sqrt{1-\sin^2 A}\\ &= \pm\sqrt{1-\left(\frac{4}{5}\right)^2}\\ &= \pm\sqrt{\frac{9}{25}}\\ &= \pm\frac{3}{5}\end{aligned}$$

But A is acute (as stated in the question), so $\cos A = \frac{3}{5}$

To find $\cos B$:

$\sin B = \frac{1}{\sqrt{2}}$ and B is acute, so $B = \frac{\pi}{4}$

Thus, $\cos B = \frac{1}{\sqrt{2}}$

Now substitute these values into the expanded formula:

$$\begin{aligned}\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\\ &= \frac{4}{5}\times\frac{1}{\sqrt{2}} - \frac{3}{5}\times\frac{1}{\sqrt{2}}\\ &= \frac{4-3}{5\sqrt{2}}\\ &= \frac{1}{5\sqrt{2}}\end{aligned}$$

12

$$\begin{aligned}\text{LHS} &= \frac{1-\cos(2A)}{1+\cos(2A)}\\ &= \frac{1-\cos(2A)}{1+\cos(2A)}\times\frac{1-\cos(2A)}{1-\cos(2A)}\\ &= \frac{(1-\cos(2A))^2}{1^2-\cos^2(2A)}\\ &= \frac{(1-\cos(2A))^2}{\sin^2(2A)}\end{aligned}$$

But $\cos(2A) = 1 - 2\sin^2(A)$

$$\begin{aligned}\text{LHS} &= \frac{(1-\cos(2A))^2}{\sin^2(2A)}\\ &= \frac{(1-\left[1-2\sin^2(A)\right])^2}{\sin^2(2A)}\\ &= \frac{(1-1+2\sin^2(A))^2}{\sin^2(2A)}\\ &= \frac{4\sin^4(A)}{\sin^2(2A)}\\ &= \frac{4\sin^4(A)}{(2\sin(A)\cos(A))^2}\\ &= \frac{4\sin^4(A)}{4\sin^2(A)\cos^2(A)}\\ &= \frac{\sin^2(A)}{\cos^2(A)}\\ &= \tan^2(A)\\ &= \text{RHS}\end{aligned}$$

Thus, LHS = RHS as required.

13 a

$$\begin{aligned}AC^2 &= 4^2 + 1^2 - 2\times 4\times 1\times\cos(120^\circ)\\ &= 16 + 1 - 8\times\frac{-1}{2}\\ &= 17 + 4\\ &= 21\\ AC &= \sqrt{21}\end{aligned}$$

b

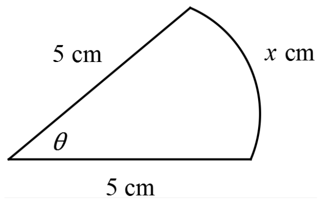
$$\begin{aligned}\frac{\sin(\angle BAC)}{4} &= \frac{\sin(\angle ABC)}{\sqrt{21}}\\ \sin(\angle BAC) &= \frac{4\sin(120^\circ)}{\sqrt{21}}\\ &= \frac{2\sqrt{3}}{\sqrt{21}}\\ &= \frac{2}{\sqrt{7}}\end{aligned}$$

c $A = \frac{1}{2} \times 1 \times 4 \times \sin(\angle ABC)$

$= \frac{1}{2} \times 1 \times 4 \times \sin(120°)$

$= \sqrt{3}\,\text{cm}^2$

14



$5 + 5 + x = 16$

$x = 6$

$x = r\theta$

$6 = 5\theta$

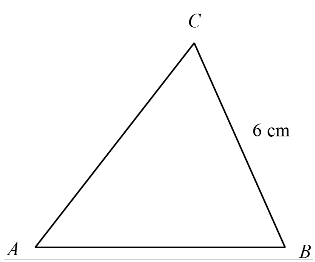
$\theta = \frac{6}{5}$

$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 5^2 \times \frac{6}{5}$

$= 15\,\text{cm}^2$

15



$\frac{AC}{\sin(B)} = \frac{6}{\sin(A)}$

$= \frac{6\sin(B)}{\sin(A)}$

But $\sin(B) = 2\sin(A)$

Thus,

$\frac{AC}{\sin(B)} = \frac{6}{\sin(A)}$

$AC = \frac{6 \times (2\sin(A))}{\sin(A)}$

$AC = 12\,\text{cm}$

16 **a** $(-2a^2)^3 \times 3a^4 = -8a^6 \times 3a^4$

$= -24a^{10}$

b $\frac{5a^4 \times 2ab^2}{20a^2b^4} = \frac{10a^5b^2}{20a^2b^4}$

$= \frac{a^3}{2b^2}$

c $\frac{(xy^{-2})^{-1}}{y} \times \frac{3x^{-1}y^2}{4(xy)^3} = \frac{x^{-1}y^2}{y} \times \frac{3x^{-1}y^2}{4x^3y^3}$

$= \frac{3y^3}{4x^5y^3}$

$= \frac{3}{4x^5}$

d

$\left(\frac{4a^2}{ab}\right)^3 \div (2ab^{-1})^3 = \left(\frac{64a^6}{a^3b^3}\right) \div (8a^3b^{-3})$

$= \frac{64a^6}{a^3b^3} \times \frac{1}{8a^3b^{-3}}$

$= \frac{64a^6}{8a^6}$

$= 8$

e $\sqrt{x^{-1}y^2} \times \left(\frac{y}{x}\right)^{-\frac{1}{3}} = x^{-\frac{1}{2}}y \times y^{-\frac{1}{3}}x^{\frac{1}{3}}$

$= x^{-\frac{1}{6}}y^{\frac{2}{3}}$

$= \frac{y^{\frac{2}{3}}}{x^{\frac{1}{6}}}$

f
$$\sqrt{2x-1}\times(2x-1)^{-1} = (2x-1)^{\frac{1}{2}}(2x-1)^{-1}$$
$$= \frac{1}{(2x-1)^{\frac{1}{2}}}$$

17 a $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^{2} = \frac{25}{9}$

b $\left(\frac{4^2}{2^6}\right)^{-2} = \frac{4^{-4}}{2^{-12}} = \frac{2^{-8}}{2^{-12}} = 2^4 = 16$

c $\frac{27^2\times 9^3}{81^2} = \frac{3^6\times 3^6}{3^8} = 3^4 = 81$

d $(-27)^{-\frac{1}{3}} = \frac{1}{-27^{\frac{1}{3}}} = -\frac{1}{3}$

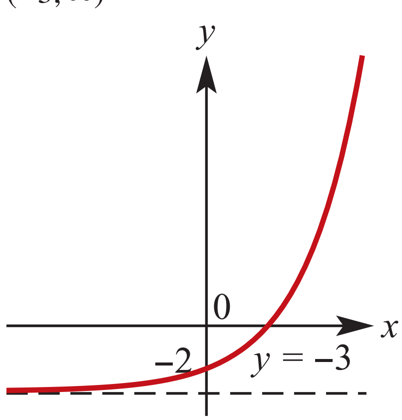
18 a $\frac{9^{2n}\times 8^n\times 16^n}{6^n} = \frac{3^{4n}\times 2^{3n}\times 2^{4n}}{3^n 2^n}$
$= 2^{6n}3^{3n}$

b $3\log_2(16) = 3\times 4 = 12$

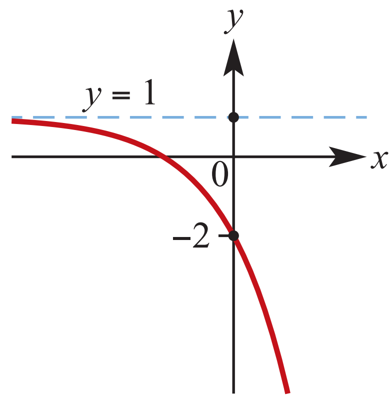
c $2\log_{10}3 + \log_{10}4 = \log_{10}(3^2\times 4)$
$= \log_{10}36$

d $\log_3\left(\frac{1}{27}\right) = \log_3(3^{-3}) = -3$

19 a $f:\mathbb{R}\to\mathbb{R},\ f(x) = 2^x - 3$ Range $(-3,\infty)$



b $f:\mathbb{R}\to\mathbb{R},\ f(x) = -3\times 2^x + 1$
Range $(-\infty, 1)$



20 a $4^x = 8^{x-1}$
$2^{2x} = 2^{3x-3}$
$2x = 3x - 3$
$x = 3$

b $4^x = 5\times 2^x - 4$
$2^{2x} - 5\times 2^x + 4 = 0$
$(2^x - 4)(2^x - 1) = 0$
$x = 2$ or $x = 0$

c $5^{x-1} > 125$
$\Leftrightarrow 5^{x-1} > 5^3$
$\Leftrightarrow x - 1 > 3$
$\Leftrightarrow x > 4$

d $\log_2(x+1) = 3$
$x + 1 = 2^3$
$x = 7$

e $\log_4(2x) - \log_4(x+1) = 0$

$$\log_4 \frac{2x}{x+1} = 0$$

$$\frac{2x}{x+1} = 4^0$$

$$2x = x + 1$$

$$x = 1$$

21 **a** $2^x = 5 \Leftrightarrow x = \log_2(5)$

b $5^{3x+1} = 10$

$$5^{3x} = 2$$

$$3x = \log_5(2)$$

$$x = \frac{1}{3}\log_5(2)$$

c $0.6^x < 0.2$

$$\Leftrightarrow x\log_{10}(0.6) < \log_{10} 0.2$$

$$\Leftrightarrow x > \frac{\log_{10} 0.2}{\log_{10}(0.6)}$$

22 When n increases by 1, t_n increases by 6. Hence, the common difference is 6.

23 $a = \frac{-3}{4}$

$$d = \frac{-11}{4} - \frac{-3}{4} = -2$$

$$S_{12} = \frac{12}{2}\left(2 \times \frac{-3}{4} + (12-1) \times (-2)\right)$$

$$= 6\left(\frac{-3}{2} - 22\right)$$

$$= -141$$

24 **a** $S_{n-1} = 2(n-1)^2 + 3(n-1)$

$$= 2(n^2 - 2n + 1) + 3n - 3$$

$$= 2n^2 - 4n + 2 + 3n - 3$$

$$= 2n^2 - n - 1$$

b $S_n = S_{n-1} + t_n$

$$2n^2 + 3n = 2n^2 - n - 1 + t_n$$

$$t_n = 4n + 1$$

c $t_1 = 4 \times 1 + 1 = 5$

d From part **b**, if n increases by 1, t_n increases by 4. The common difference is 4.

25 From the pattern, it can be seen that $t_{n+1} = 2t_n$.
This is a geometric sequence with first term 3 and common ratio 2.
Let $S_n = 189$:

$$3 \times \frac{1 - 2^n}{1 - 2} = 189$$

$$2^n = 64$$

$$= 2^6$$

$$n = 6$$

26 $a = 4,\ d = \frac{1}{2}$

$$S = \frac{4}{1 - \frac{1}{2}}$$

$$= 8\text{ m}$$

The frog travels 8 metres.

27 The side lengths are: $\frac{36}{r^2}$, $\frac{36}{r}$ and 36 cm.

$$\frac{36}{r^2} + \frac{36}{r} + 36 = 76$$

$$\frac{36}{r^2} + \frac{36}{r} - 40 = 0$$

$$36 + 36r - 40r^2 = 0$$

$$40r^2 - 36r - 36 = 0$$

$$10r^2 - 9r - 9 = 0$$

$$(2r - 3)(5r + 3) = 0$$

$$r = \frac{3}{2} \text{ as } r > 0$$

Shortest side is $\frac{36}{\left(\frac{3}{2}\right)^2} = 16$ cm

28 **a** First even number is 2.
Common difference is 2 as consecutive even numbers differ by 2.
There are 50 even numbers in the first 100 natural numbers.

$$S_{50} = \frac{50}{2}(2 \times 2 + (50 - 1) \times 2)$$
$$= 2550$$

b This can be found by firstly finding the sum of the first 100 natural numbers, and then subtracting all the numbers that *are* divisible by 3.
Sum of first 100 natural numbers:

$$S_{100} = \frac{100}{2}(2 \times 1 + (100 - 1) \times 1)$$
$$= 5050$$

Sum of all numbers that are divisible by 3:
First term = 3
Last term = 99, which is the 33^{rd} term.
Common difference = 3, because consecutive numbers that are divisible by 3 will differ by 3.

$$S_{33} = \frac{33}{2}(2 \times 3 + (33 - 1) \times 3)$$
$$= 1683$$

Hence, the required sum is
5050 − 1683 = 3367.

Solutions to 16B Multiple-choice questions

1 D $2x = 2x\left(\frac{180}{\pi}\right)^\circ = \left(\frac{360x}{\pi}\right)^\circ = \frac{360x^\circ}{\pi}$

2 A $y = \sin 2x + 1$
Q is at the 1st maximum:
$x = \frac{\pi}{4}$, $y = \sin\frac{\pi}{2} + 1 = 2$

3 D $1 - 3\cos\theta$
range $= [1-3, 1+3] = [-2, 4]$,
so min value $= -2$

4 D $y = 16 + 15\sin\frac{\pi x}{60}$
$\therefore\ y(10) = 16 + 15\sin\frac{10\pi}{60} = 16 + \frac{15}{2} = 23.5$ m

5 D $\sin(\pi+\theta) + \cos(\pi+\theta) = -\sin\theta - \cos\theta$

6 A $\sin x = 0$, $\therefore\ x = 0,\ \pi$
Over $[0,\ \pi]$, **B**, **C**, **E** have 1 solution and **D** has none.

7 E $y = \sin\frac{\theta}{2}$ has per $= 4\pi$

8 D $2 - 3\sin\theta$
range $= [2-3, 2+3] = [-1, 5]$

9 D $y = \cos x^\circ$ with translation of 30° in negative x direction
$\therefore\ y = \cos(x+30)^\circ$

10 E $f(x) = -2\cos 3x$:
per $\frac{2\pi}{3}$, ampl 2

11 E We first must find $\cos A$ and $\cos B$. Since both angle are acute, we know that $\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$, $\cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\left(\frac{8}{17}\right)^2} = \frac{15}{17}$.
Therefore,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12},$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}.$$

Therefore,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{5}{12} + \frac{8}{15}}{1 - \frac{5}{12}\frac{8}{15}} = -\frac{171}{140}.$$

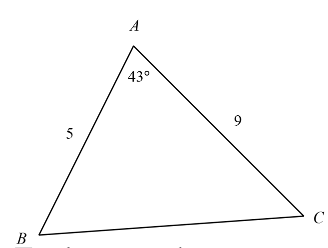
12 B Since

$$\frac{c}{\sin 38^\circ} = \frac{58}{\sin 130^\circ}$$

it follow that

$$c = \frac{58\sin 38^\circ}{\sin 130^\circ}.$$

13 D



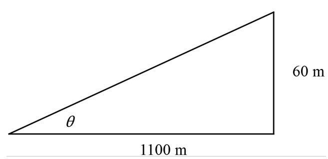
Evaluate each statement carefully:

I. Yes, because two side lengths and the included angle is known.
II. Yes, using the sine rule.
III. Yes, using the cosine rule.

14 A Since angle A is the angle between the given sides, the area will be given by

$$A = \frac{1}{2} \times 6 \times 7 \sin 48°.$$

15 B



$$\tan\theta = \frac{60}{1100}$$
$$\theta = \tan^{-1}\frac{60}{1100}$$
$$\approx 3.12°$$

16 E $l = 3,\ r = 4$

$$l = r\theta$$
$$3 = 4\theta$$
$$\theta = \frac{3}{4}$$
$$= \frac{3}{4} \times \frac{180°}{\pi}$$
$$\approx 43°$$

17 C Diameter of 10 cm means radius of 5 cm.
Use radian angle in area formula:
$60° = \frac{\pi}{3}$

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 5^2 \times \frac{\pi}{3}$$
$$\approx 13.09$$

18 B We simply find the area of the circle segment,

$$A = \frac{\pi r^2\theta}{360} - \frac{1}{2}r^2 \sin\theta$$
$$= \frac{\pi \times 45^2 \times 110}{360} - \frac{1}{2} \times 45^2 \sin 110°$$
$$\approx 992 \text{ cm}^2$$

19 E

$$\frac{\sqrt{1.21 \times 10^{-6}}}{2 \times 10^{-4}} = \frac{1.1 \times 10^{-3}}{2 \times 10^{-4}}$$
$$= 0.55 \times 10^{-3-(-4)}$$
$$= 0.55 \times 10$$
$$= 5.5$$

20 B $\log_a 8 = 3,\ \therefore a^3 = 8$

$$a = 2$$

21 B

$$5^{n-1}5^{n+1} = 5^{n-1+n+1}$$
$$= 5^{2n}$$

22 B $2^x = \frac{1}{64},\ \therefore 2^x = 2^{-6}$

$$\therefore\ x = -6$$

23 E

$$125^a 5^b = 5^{3a}5^b$$
$$= 5^{3a+b}$$

24 D

$$4^x = 10 - 4^{x+1}$$
$$\therefore\ 4^x + 4^{x+1} = 10$$
$$4^x(1 + 4) = 10$$
$$5(4^x) = 10$$
$$4^x = 2 = 4^{0.5}$$
$$\therefore x = 0.5$$

25 A

$$\frac{7^{n+2} - 35(7^{n+1})}{44(7^{n+2})} = \frac{7^{n+2} - 5(7^n)}{44(7^{n+2})}$$
$$= \frac{7^n(49 - 5)}{44(7^{n+2})}$$
$$= \frac{7^n}{7^{n+2}} = \frac{1}{49}$$

26 D $f(x) = 2 + 3^x$

$$\therefore\ f(2x) - f(x) = (2 + 3^{2x}) - (2 + 3^x)$$
$$= 3^{2x} - 3^x$$
$$= 3^x(3^x - 1)$$

27 C $(7^{2x})(49^{2x-1}) = 1$

$$\therefore\ 7^{2x}7^{4x-2} = 1$$
$$7^{6x-2} = 1 = 7^\circ$$
$$\therefore\ 6x - 2 = 0, \therefore\ x = \frac{1}{3}$$

28 B $y = 2^x$ and; $y = \left(\frac{1}{2}\right)^x$

y-intercept at (0, 1)

29 A $f(x) = (2x)^0 + x^{-\frac{2}{3}}$

$$= 1 + x^{-\frac{2}{3}}$$
$$\therefore\ f(8) = 1 + 8^{-\frac{2}{3}}$$
$$= 1 + \frac{1}{4} = \frac{5}{4}$$

30 D $t_4 = a + (4 - 1)d$

$$= 4 + (4 - 1) \times 3$$
$$= 13$$

31 B $a = 5,\ d = 2$

$$t_9 = a + (9 - 1)d$$
$$= 5 + (9 - 1) \times 2$$
$$= 21$$

32 A The difference between terms is constant.

$$(y - 1) - y = (2y - 1) - (y - 1)$$
$$y - 1 - y = 2y - 1 - y + 1$$
$$-1 = y$$
$$y = -1$$

33 A $t_4 = a + 3d$

$$= 3 + 3d = 9$$
$$3d = 6$$
$$d = 2$$
$$t_{11} = a + (n - 1)d$$
$$= 3 + 10 \times 2$$
$$= 23$$

34 D $a = \frac{1}{2},\ r = -\frac{1}{2}$

$$S_\infty = \frac{a}{1 - r}$$
$$= \frac{\frac{1}{2}}{1 - -\frac{1}{2}}$$
$$= \frac{\frac{1}{2}}{\frac{3}{2}}$$
$$= \frac{1}{3}$$

35 C $\frac{a}{1 - r} = 4a$

Multiply both sides by $\frac{1 - r}{a}$.

$$1 = 4(1 - r)$$
$$1 = 4 - 4r$$
$$4r = 4 - 1$$
$$r = \frac{3}{4}$$

36 C $a = 1,\ r = -3x$

$$S_n = a\frac{1 - r^n}{1 - r}$$
$$= \frac{1 - (-3x)^n}{1 - (-3x)}$$
$$= \frac{1 - (-3x)^n}{1 + (-3x)}$$
$$= \frac{(-3x)^n - 1}{-1 - (-3x)}$$

37 A Split the sum into two components:

$1-2+3-4+5-6+\dots$

$=(1+3+5+\dots)-(2+4+6+\dots)$

For the first sum: $a=1, d=2$ and $n=1000$

$$S_{1000}=\frac{1000}{2}(1\times 2+(1000-1)\times 2)$$
$$=10\,000\,000$$

For the second sum: $a=2, d=2$ and $n=1000$

$$S_{1000}=\frac{1000}{2}(2\times 2+(1000-1)\times 2)$$
$$=10\,010\,000$$

Thus,

$1-2+3-4+5-6+\dots$

$=(1+3+5+\dots)-(2+4+6+\dots)$

$=10\,000\,000-10\,001\,000$

$=-1000$

38 D $\cos\theta-\sin\theta=\dfrac{1}{4}$

$$\therefore\ (\cos\theta-\sin\theta)^2=\frac{1}{16}$$
$$\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta=\frac{1}{16}$$
$$1-2\sin\theta\cos\theta=\frac{1}{16}$$
$$2\sin\theta\cos\theta=1-\frac{1}{16}$$
$$\therefore\ \sin\theta\cos\theta=\frac{15}{32}$$

39 B $y=\dfrac{1}{2}\sin 2x$ and $y=\dfrac{1}{2}$ meet at

$\sin 2x=1$

$$\therefore 2x=\frac{\pi}{2},\frac{5\pi}{2}\dots$$
$$x=\frac{\pi}{4},\frac{5\pi}{4}\dots$$

40 D We have,

$$\cos A\cos B-\sin A\sin B=\cos(A+B)$$
$$=\cos\frac{\pi}{2}$$
$$=0.$$

41 E Considering right $\triangle VOE$, we have

$$\tan\theta=\frac{VO}{OE}$$
$$=\frac{100}{40}$$
$$=\frac{5}{2},$$

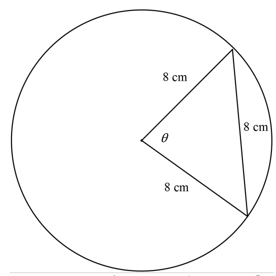
so that $\theta=\tan^{-1}\dfrac{5}{2}\approx 68°$.

42 B $\angle ABC=60°+60°=120°$

Use cosine rule:

$$AC^2=4^2+6^2-2\times 4\times 6\times\cos(120°)$$
$$AC=\sqrt{4^2+6^2-2\times 4\times 6\times\cos(120°)}$$
$$=\sqrt{4^2+6^2-48\cos(120°)}$$

43 B



Use cosine rule to find angle:

$$8^2=8^2+8^2-2\times 8\times 8\times\cos(\theta)$$
$$\cos(\theta)=\frac{1}{2}$$
$$\theta=\frac{\pi}{3}$$

Now, to find the area:

$$A=\frac{1}{2}r^2(\theta-\sin\theta)$$
$$=\frac{1}{2}\times 8^2\times(\frac{\pi}{3}-\sin\frac{\pi}{3})$$
$$=32(\frac{\pi}{3}-\frac{\sqrt{3}}{2})$$

44 A

$$C^d=3$$
$$\therefore C^{4d}-5=3^4-5$$
$$=76$$

45 E $\log_2 56 - \log_2 7 + \log_2 2$

$= \log_2\left(\frac{56 \times 2}{7}\right)$

$= \log_2 16$

$= 4$

46 B $\log_b a = c;\ \log_x b = c$

$\therefore\ a = b^c,\ b = x^c$

$\therefore\ \log_a b = c \log_a x$

$\therefore \log_a x = \frac{1}{c} \log_a b = \frac{1}{c^2} \log_a b^c$

$\therefore \log_a x = \frac{1}{c^2} \log_a a = \frac{1}{c^2}$

47 B $a = S_1 = 2^2 - 2 = 2$

$S_2 = 2^3 - 2 = 6$

$t_2 = S_2 - S_1 = 4$

$r = \frac{t_2}{t_1} = 2$

$t_n = ar^{n-1}$

$= 2 \times 2^{n-1}$

$= 2^n$

48 C $0.\dot{7}\dot{2} = 0.727272\ \ldots$

$0.\dot{7}\dot{2} \times 100 = 72.7272\ \ldots$

$0.\dot{7}\dot{2} \times 99 = 72$

$0.\dot{7}\dot{2} = \frac{72}{99}$

49 A $0.\dot{3}\dot{6} = 0.363636\ \ldots$

$0.\dot{3}\dot{6} \times 100 = 36.3636\ \ldots$

$0.\dot{3}\dot{6} \times 99 = 36$

$0.\dot{3}\dot{6} = \frac{36}{99} = \frac{4}{11}$

Numerator + denominator = 4 + 11

= 15

50 $t_5 = a + (5 - 1)d = 1.6$

$t_{12} = a + (12 - 1)d = -1.9$

Solve these equations simultaneously on the CAS calculator to get:

$a = 3.6$ and $d = -0.5$

Thus,

$t_{15} = a + (15 - 1)d$

$= 3.6 + 14 \times (-0.5)$

$= -3.4$

51 C There are 8 terms, $a = -4$ and $t_8 = 10$.

$a + 7d = 10$

$-4 + 7d = 10$

$7d = 14$

$d = 2$

The required sum is $S_7 - a$.

$S_7 - a = \frac{7}{2}(-8 + 6 \times 2) - -4$

$= 14 + 4$

$= 18$

Solutions to 16C Extended-response questions

1 a $\angle ADB = 180° - (50 + 30)° - 28° = 72°$

Use sine rule:

$$\frac{AD}{\sin(28°)} = \frac{100}{\sin(72°)}$$

$$AD \approx 49\text{ m}$$

$\angle ACB = 180° - (22 + 28)° - 30° = 100°$

Use sine rule:

$$\frac{AC}{\sin(22° + 28°)} = \frac{100}{\sin(100°)}$$

$$AC \approx 78\text{ m}$$

b $AD = 49.363161$

$AC = 77.786191$

$\angle DAC = 50°$

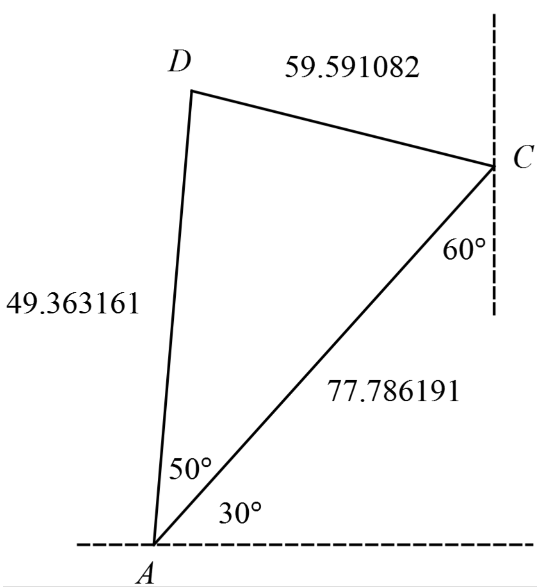
Use cosine rule:

$$DC = \sqrt{\begin{array}{l}49.363161^2 + 77.786191^2 \\ -2 \times 49.363161 \times 77.786191 \times \cos(50°)\end{array}}$$

$$= 59.591082$$

The distance between the two platforms is 60 metres, to the nearest metre.

c Consider the diagram below:



Find $\angle ACD$ using sine rule:

$$\frac{\sin(\angle ACD)}{49.363161} = \frac{\sin(50°)}{59.591082}$$

$$\angle ACD = \sin^{-1}\left(\frac{49.363161 \times \sin(50°)}{50.591082}\right)$$

$$= 39.387679°$$

Let the angle that the line DC makes with the reference North line to be θ:

$$180° = 60° + 39.387679° + \theta$$

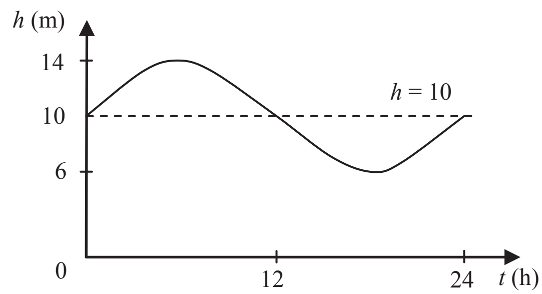
$$\theta = 80.612321°$$

Thus, the bearing is $360° - \theta \approx 279°\text{T}$

2 a $h(t) = 10 + 4\sin(15t)°,\ 0 \le t \le 24$

period = $\frac{360}{15}$ = 24, amplitude = 4

translation of 10 units in the positive direction of the h-axis



b When $h = 13$, $10 + 4\sin(15t) = 13$

$\therefore\quad 4\sin(15t) = 3 \qquad \therefore\quad \sin(15t) = \frac{3}{4}$

$\therefore\quad 15t = \sin^{-1}\left(\frac{3}{4}\right) \quad$ or $\quad 15t = 180 - \sin^{-1}\left(\frac{3}{4}\right)$

and $\quad t = \frac{1}{15}\sin^{-1}\left(\frac{3}{4}\right) \quad$ or $\quad t = \frac{1}{15}\left(180 - \sin^{-1}\left(\frac{3}{4}\right)\right)$

From the graph it can be seen that only two solutions are required.

$\therefore\quad t \approx \frac{1}{15}(48.5904) \quad$ or $\quad t \approx \frac{1}{15}(180 - 48.5904)$

$\approx 3.2394 \qquad \approx 8.7606$

Hence, $h = 13$ after approximately 3.2394 hours and 8.7606 hours.

c When $h = 11$, $10 + 4\sin(15t) = 11$

$\therefore\quad 4\sin(15t) = 1 \qquad \therefore\quad \sin(15t) = \frac{1}{4}$

$\therefore\quad 15t = \sin^{-1}(0.25) \quad$ or $\quad 15t = 180 - \sin^{-1}(0.25)$

and $\quad t = \frac{1}{15}\sin^{-1}(0.25) \quad$ or $\quad t = \frac{1}{15}(180 - \sin^{-1}(0.25))$

From the graph only two solutions are required for the domain $0 \le t \le 24$.

$\therefore \quad t \approx \frac{1}{15}(14.4775) \quad \text{or} \quad t \approx \frac{1}{15}(180 - 14.4775)$

$\approx 0.9652 \qquad \approx 11.0348$

For $h \geq 11, 0.9652 \leq t \leq 11.0348$ (approximately).

Hence a boat can leave the harbour between 0.9652 hours and 11.0348 hours.

3 a At the start of the experiment, $t = 0$.

$$\begin{aligned}\therefore N(0) &= 40 \times 2^{1.5(0)} \\ &= 40 \times 2^0 \\ &= 40 \times 1 = 40\end{aligned}$$

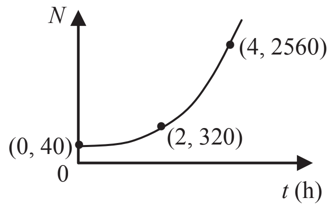
Hence there are 40 bacteria present at the start of the experiment.

b i When $t = 2$,
$$\begin{aligned}N(2) &= 40 \times 2^{1.5(2)} \\ &= 40 \times 2^3 \\ &= 40 \times 8 \\ &= 320\end{aligned}$$
After 2 hours, there are 320 bacteria present.

ii When $t = 4$,
$$\begin{aligned}N(4) &= 40 \times 2^{1.5(4)} \\ &= 40 \times 2^6 \\ &= 40 \times 64 \\ &= 2560\end{aligned}$$
After 4 hours, there are 2560 bacteria present.

iii When $t = 12$,
$$\begin{aligned}N(12) &= 40 \times 2^{1.5(12)} \\ &= 40 \times 2^{18} \\ &= 40 \times 262\,144 \\ &= 10\,485\,760\end{aligned}$$
After 12 hours, there are 10 485 760 bacteria present.

c



d When $N = 80$, $80 = 40 \times 2^{1.5(t)}$

$\therefore \quad 2^{1.5(t)} = 2^1$

$\therefore \quad 1.5t = 1$

$\therefore \quad t = \frac{2}{3}$

The number of bacteria doubles after $\frac{2}{3}$ of an hour (40 minutes).

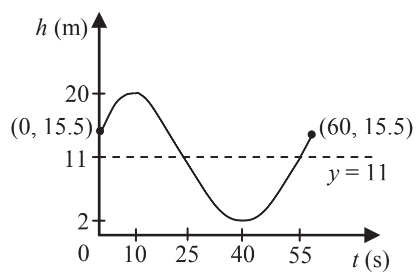
4 **a** The Ferris wheel makes one revolution after one period.

$$\text{Period} = \frac{2\pi}{n}, \text{ where } n = \frac{\pi}{30}$$
$$= 2\pi \div \frac{\pi}{30}$$
$$= \frac{2\pi \times 30}{\pi}$$
$$= 60$$

i.e. the Ferris wheel takes 60 seconds for one revolution.

b Period = 60, amplitude = 9

The graph is translated 10 units in the positive direction of the t-axis and 11 units in the positive direction of the h-axis.



c Range = [2, 20]

d At $h = 2$, $11 + 9\cos\left(\frac{\pi}{30}(t-10)\right) = 2$

$\therefore \quad 9\cos\left(\frac{\pi}{30}(t-10)\right) = -9$

$\therefore \quad \cos\left(\frac{\pi}{30}(t-10)\right) = -1$

$\therefore \quad \frac{\pi}{30}(t-10) = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \ldots$

$\therefore \quad t - 10 = 30 \text{ or } 90 \text{ or } 150 \text{ or } \ldots$

$\therefore \quad t = 40 \text{ or } 100 \text{ or } 160 \text{ or } \ldots$

i.e. the height of the person above the ground is 2 m after 40 seconds and then again after each further 60 seconds.

e At $h = 15.5$, $11 + 9\cos\left(\frac{\pi}{30}(t-10)\right) = 15.5$

$\therefore$ $9\cos\left(\frac{\pi}{30}(t-10)\right) = 4.5$

$\therefore$ $\cos\left(\frac{\pi}{30}(t-10)\right) = \frac{1}{2}$

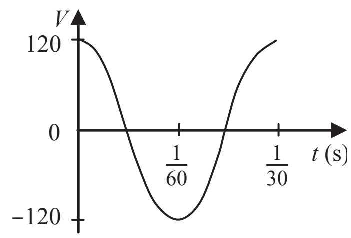
$\therefore$ $\frac{\pi}{30}(t-10) = \frac{-\pi}{3}$ or $\frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$ or ...

$\therefore$ $t - 10 = -10$ or 20 or 50 or 70 or ...

$\therefore$ $t = 0$ or 20 or 60 or 80 or ...

i.e. the height of the person above the ground is 15.5 m at the start and each 60 seconds thereafter, and also at 20 seconds and each 60 seconds after that.

5 a $V = 120\cos(60\pi t)$, period $= \frac{2\pi}{60\pi} = \frac{1}{30}$, amplitude $= 120$



b At $V = 60$, $120\cos(60\pi t) = 60$

$\therefore$ $\cos(60\pi t) = \frac{1}{2}$

$\therefore$ $60\pi t = \frac{\pi}{3}$ (Only smallest positive solution is required.)

$\therefore$ $t = \frac{\pi}{3 \times 60\pi}$

$= \frac{1}{180}$

i.e. the first time the voltage is 60 is at $\frac{1}{180}$ second.

c The voltage is maximised when $V = 120$

$\therefore$ $120\cos(60\pi t) = 120$

$\therefore$ $\cos(60\pi t) = 1$

$\therefore$ $60\pi t = 0$ or 2π or 4π or ...

$\therefore$ $t = \frac{0}{60\pi}$ or $\frac{2\pi}{60\pi}$ or $\frac{4\pi}{60\pi}$ or ...

$= 0$ or $\frac{1}{30}$ or $\frac{1}{15}$ or ...

i.e. the voltage is maximised when $t = 0$ seconds, and every $\frac{1}{30}$ second thereafer ($t = \frac{k}{30}, k = 0, 1, 2, \dots$).

6 $d = a + b \sin c(t - h)$

a **i** $\text{period} = \dfrac{60 \text{ seconds}}{4 \text{ revolutions}}$

$= 15 \text{ seconds}$

ii amplitude = radius of waterwheel

$= 3$ metres

iii $\text{period} = \dfrac{2\pi}{c} = 15$

$\therefore \quad c = \dfrac{2\pi}{15}$

b At $(0, 0)$, $\quad 0 = a + b \sin\left(\dfrac{2\pi}{15}(0 - h)\right)$

Now amplitude = 3, $\quad \therefore b = 3$

and the translation in the positive direction of the y-axis is 2,

$\therefore \quad a = 2$

$\therefore \quad 0 = 2 + 3 \sin \dfrac{-2\pi h}{15}$

$\therefore \quad 3 \sin \dfrac{-2\pi h}{15} = -2$

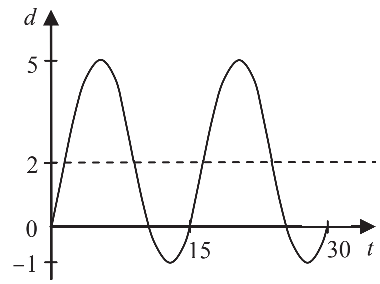
$\therefore \quad \sin \dfrac{-2\pi d}{15} = \dfrac{-2}{3}$

$\therefore \quad \dfrac{-2\pi h}{15} \approx -0.729\,727\,656$

$\therefore \quad h \approx \dfrac{-0.729\,727\,656 \times 15}{-2\pi}$

$\approx 1.742\,10$

c $d = 2 + 3\sin\left(\frac{2\pi}{15}(t - 1.742\,10)\right)$



7 a i When $t = 0$, $h = 30(1.65)^0$

$= 30 \times 1$

$= 30$

ii When $t = 1$, $h = 30(1.65)^1$

$= 30 \times 1.65$

$= 49.5$

iii When $t = 2$, $h = 30(1.65)^2$

$= 30 \times 2.7225$

$= 81.675$

b

$$\begin{aligned} h(N) &= 30(1.65)^N \\ h(N+1) &= 30(1.65)^{N+1} \\ &= 30(1.65)^N \times 1.65 \\ &= 1.65h(N) \\ \therefore \quad h(N+1) &= kh(N) \end{aligned}$$

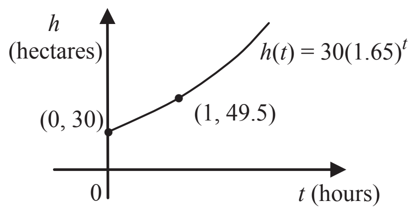
implies $k = 1.65$

c When $h = 900$, $30(1.65)^t = 900$

$$\begin{aligned} \therefore \quad 1.65^t &= 30 \\ \therefore \quad \log_{10} 1.65^t &= \log_{10} 30 \\ \therefore \quad t \log_{10} 1.65 &= \log_{10} 30 \\ \therefore \quad t &= \frac{\log_{10} 30}{\log_{10} 1.65} \\ &\approx 6.792 \end{aligned}$$

i.e. it takes approximately 6.792 hours for 900 hectares to be burnt.

d



8 a $P_1 = 4 \times 1 = 4$

b $P_2 = 3 \times 1 + 6 \times \dfrac{1}{2}$

$= 3 + 3$

$= 6$

c $P_3 = 3 \times 1 + 5 \times \dfrac{1}{2} + 10 \times \dfrac{1}{4}$

$= 3 + \dfrac{5}{2} + \dfrac{5}{2}$

$= 8$

d The common difference is 2 as $8 - 6 = 2$ and $6 - 4 = 2$.

e i $P_4 = P_3 + 2$

$= 8 + 2$

$= 10$

ii $P_n = P_{n-1} + 2$

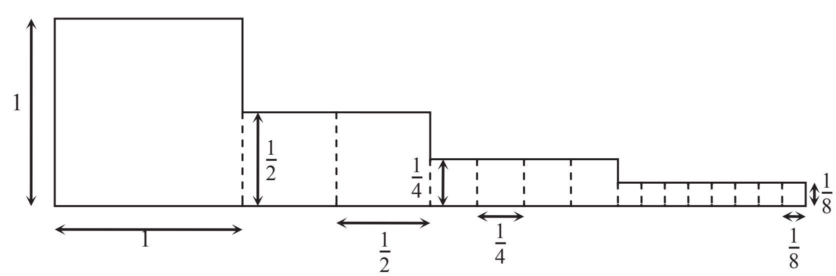
iii $P_n = P_1 + (n-1) \times 2$

$= 4 + 2(n-1)$

$= 4 + 2n - 2$

$= 2n + 2$

iv



9 a When $t = 0$, $\theta = 80(2^{-0}) + 20$

$= 80 + 20$

$= 100$

When $t = 1$, $\theta = 80(2^{-1}) + 20$

$= 40 + 20$

$= 60$

When $t = 2$, $\theta = 80(2^{-2}) + 20$

$= 20 + 20$

$= 40$

When $t = 3$, $\theta = 80(2^{-3}) + 20$

$= 10 + 20$

$= 30$

When $t = 4$, $\theta = 80(2^{-4}) + 20$

$= 5 + 20$

$= 25$

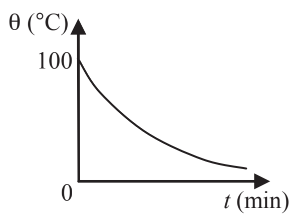
When $t = 5$, $\theta = 80(2^{-5}) + 20$

$= 2.5 + 20$

$= 22.5$

t	0	1	2	3	4	5
θ	100	60	40	30	25	22.5

b



c When $\theta = 60°$, $t = 1$
i.e. the temperature is 60°C after 1 minute.

d When $t = 3.5$, $\theta = 80(2^{-3.5}) + 20$

$$\approx \frac{80}{11.313\,708\,5} + 20$$

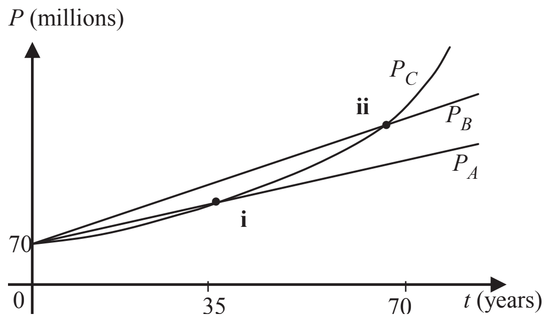
≈ 27.071

10 **a** $P_A = 70\,000\,000 + 3\,000\,000t,$

$P_B = 70\,000\,000 + 5\,000\,000t$

$P_C = 70\,000\,000(1.3)^{\frac{t}{10}}$

b



c From the graph, the population of C overtakes the population of

i A after approximately 35 years

ii B after approximately 67 years.

11 **a** **i** When $t = 1975$,
$$\begin{aligned} P &= 4(2)^{\frac{1975-1975}{35}} \\ &= 4(2)^0 \\ &= 4 \times 1 \\ &= 4 \text{ billion} \end{aligned}$$

ii When $t = 1995$,
$$\begin{aligned} P &= 4(2)^{\frac{1995-1975}{35}} \\ &= 4(2)^{\frac{20}{35}} \\ &\approx 4 \times 1.485\,99 \\ &\approx 5.944 \text{ billion} \end{aligned}$$

iii When $t = 2005$,
$$\begin{aligned} P &= 4(2)^{\frac{2005-1975}{35}} \\ &= 4(2)^{\frac{30}{35}} \\ &\approx 7.246 \text{ billion} \end{aligned}$$

b When $t = 1997$,
$$\begin{aligned} P &= 4(2)^{\frac{1997-1975}{35}} \\ &= 4(2)^{\frac{22}{35}} \end{aligned}$$

In 1997, $P = 4(2)^{\frac{1997-1975}{35}}$

$= 4(2)^{\frac{22}{35}}$

Double this is $2 \times 4(2)^{\frac{22}{35}}$

$= 4(2)^{1+\frac{22}{35}}$

$= 4(2)^{\frac{57}{35}}$

Solve for t: $4(2)^{\frac{t-1975}{35}} = 4(2)^{\frac{57}{35}}$

Then $t - 1975 = 57$

$t = 2032$

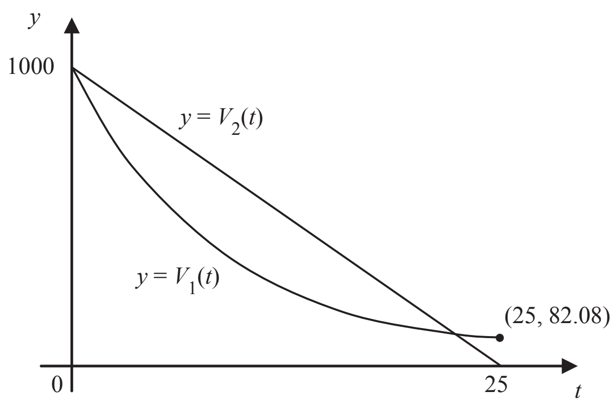
i.e. in 2032 the population of Earth will be twice the population it was in 1997.

12 $V_1(t) = 1000e^{\frac{-t}{10}}, \quad t \geq 0$

$V_2(t) = 1000 - 40t, \quad 0 \leq t \leq 25$

a $V_1(0) = 1000,\ V_2(0) = 1000$

b



c Tank B is empty when $t = 25$, i.e. when $1000 - 40t = 0$.

$V_1(25) = 10003^{\frac{-25}{10}}$

$= 64.15\ldots$

Tank A has 64.15 litres in it when B is first empty.

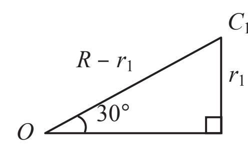
d On a CAS calculator, with **f1 = 10003^(−x/10)** and **f2 = 1000 − 40x**

$t = 0$, and $V_1(0) = V_2(0) = 1000$

$t = 23$.

13 a i $OC_1 = R - r_1$

ii $$\sin 30° = \frac{1}{2}$$



$$\text{and } \sin 30° = \frac{r_1}{R - r_1} = \frac{r_1}{OC_1}$$

$$\therefore \quad \frac{r_1}{OC_1} = \frac{1}{2}$$

$$\therefore \quad \frac{r_1}{R - r_1} = \frac{1}{2}$$

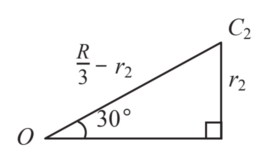
$$\therefore \quad 2r_1 = R - r_1$$

$$\therefore \quad 3r_1 = R$$

$$\therefore \quad r_1 = \frac{R}{3}$$

b i $$OC_2 = (R - 2r_1) - r_2$$
$$= R - 2 \times \frac{R}{3} - r_2$$
$$= \frac{R}{3} - r_2$$

ii $$\sin 30° = \frac{1}{2}$$



$$\text{and } \sin 30° = \frac{r_2}{\frac{R}{3} - r_2}$$

$$\therefore \quad \frac{r_2}{\frac{R}{3} - r_2} = \frac{1}{2}$$

$$\therefore \quad 2r_2 = \frac{R}{3} - r_2$$

$$\therefore \quad 3r_2 = \frac{R}{3}$$

$$\therefore \quad r_2 = \frac{R}{9}$$

c **i** The common ratio is
$$r = \frac{r_2}{r_1} = \frac{R}{9} \div \frac{R}{3} = \frac{R}{9} \times \frac{3}{R} = \frac{1}{3}$$

ii
$$r_1 = \frac{R}{3}$$
and $r_2 = \frac{R}{9} = \frac{R}{3^2}$
$$\therefore \quad r_n = \frac{R}{3^n}$$

iii
$$S_\infty = \frac{a}{1-r} = \frac{\frac{R}{3}}{1-\frac{1}{3}} = \frac{R}{3} \div \frac{2}{3} = \frac{R}{3} \times \frac{3}{2} = \frac{R}{2}$$
The sum to infinity is $\frac{R}{2}$.

iv Let A_n be the area of the circle with radius r_n.
$$\therefore \quad A_n = \pi r_n^2$$
$$\therefore \quad A_1 = \pi r_1^2 = \pi\left(\frac{R}{3}\right)^2 = \frac{\pi R^2}{9}$$
and
$$A_2 = \pi r_2^2 = \pi\left(\frac{R}{9}\right)^2 = \frac{\pi R^2}{81}$$

The common ratio is
$$r = \frac{A_2}{A_1} = \frac{\pi R^2}{81} \div \frac{\pi R^2}{9} = \frac{\pi R^2}{81} \times \frac{9}{\pi R^2} = \frac{1}{9}$$

$$S_\infty = \frac{a}{1-r} = \frac{\pi R^2}{9} \div \left(1 - \frac{1}{9}\right) = \frac{\pi R^2}{9} \times \frac{9}{8} = \frac{\pi R^2}{8}$$

The sum to infinity of the area of the circles with radii $r_1,\ r_2,\ r_3,\ \ldots$ is $\frac{\pi R^2}{8}$ square units.

14 a i Production of Company A in nth month
$$= 1000 + 80(n-1) = 1000 + 80n - 80 = 80n + 920$$

ii Production of Company A in 24th month $= 920 + 80 \times 24 = 2840$

Production of Company B in 24th month $= 1000 \times 1.04^{23} = 2464.71554\ldots$

Company A and Company B produced 2840 and 2465 tonnes respectively, to the nearest tonne, in December 2014.

iii For Company A, the total production over n months is

$S_n = \frac{n}{2}(2a + (n-1)d)$ where $a = 1000$ and $d = 80$

$$= \frac{n}{2}(2000 + 80(n-1))$$
$$= \frac{n}{2}(2000 + 80n - 80)$$
$$= \frac{n}{2}(80n + 1920)$$
$$= 40n^2 + 960n$$
$$= 40n(n + 24)$$

iv For Company A, $S_{24} = (40 \times 24)(24 + 24) = 46\,080$

For Company B, $S_n = \dfrac{a(r^n - 1)}{r - 1}$ where $a = 1000$ and $r = 1.04$

$$\therefore \quad S_{24} = \frac{1000(1.04^{24} - 1)}{1.04 - 1} = 39\,082.604\,12$$

The total production for the period January 2013 to December 2014 inclusive, of Company A and Company B, is 46 080 and 39 083 tonnes respectively, to the nearest tonne.

b Find n for which $S_n > 100\,000$ for Company A,

$$\therefore \quad 40n(n + 24) > 100\,000$$

$$\therefore \quad 40n^2 + 960n - 100\,000 > 0$$

$$\therefore \quad n^2 + 24n - 2500 > 0$$

When $n = 39$,

$39^2 + 24 \times 39 - 2500 = -43 < 0$

When $n = 40$,

$40^2 + 24 \times 40 - 2500 = 60 > 0$

The 40th month represents Apri1 2016.

The total production of Company A passes 100 000 tonnes in Apri1 2016.

15 a Distance $= 0.5 + 9 \times 1.5$

$= 0.5 + 13.5$

$= 14$

The distance between the fence and the tenth row of carrots is 14 metres.

b $t_n = 0.5 + (n - 1) \times 1.5$

$= 0.5 + 1.5n - 1.5$

$= 1.5n - 1$

c $1.5n - 1 < 80$

$\therefore \quad 1.5n < 81$

$\therefore \quad n < \dfrac{81}{1.5}$

$\therefore \quad n < 54$

The largest number of rows possible is 53.

d Distance run by rabbit $= 2 \times 0.5 + 2 \times (0.5 + 1.5) + 2 \times (0.5 + 2 \times 1.5) + \cdots + 2 \times (0.5 + 14 \times 1.5)$

$$= 2(0.5 + (0.5 + 1.5) + (0.5 + 2 \times 1.5) + \cdots + (0.5 + 14 \times 1.5))$$

$$= 2\left(\frac{15}{2}(2 \times 0.5 + (15 - 1) \times 1.5)\right)$$

$$= 15(1 + 21)$$

$$= 330$$

The shortest distance the rabbit has to run is 330 metres.

16 a i t_7 denotes the grain production in 1992.

$$t_7 = 10 + (7 - 1) \times 0.9$$
$$= 15.4$$

The grain production in 2002 was 15.4 million tonnes.

ii t_{14} denotes the grain production in 1999.

$$t_{14} = 10 + (14 - 1) \times 0.9$$
$$= 21.7$$

The grain production in 1999 was 21.7 million tonnes.

b $t_n = a + (n - 1)d$

$$= 10 + (n - 1) \times 0.9$$
$$= 10 + 0.9n - 0.9$$
$$= 0.9n + 9.1$$

c $S_n = \frac{n}{2}(2a + (n - 1)d)$

$$\therefore \quad S_{20} = \frac{20}{2}(2 \times 10 + (20 - 1) \times 0.9)$$
$$= 10(20 + 19 \times 0.9)$$
$$= 371$$

The total grain production for the 20 years starting 1996 is 371 million tonnes.

d Let $t_n \geq 2t_1$

$$\therefore \quad 0.9n + 9.1 \geq 2 \times 10$$
$$\therefore \quad 0.9n \geq 10.9$$
$$\therefore \quad n \geq 12.111\,11\ldots$$

It takes 12.1 years for the grain production to double.

e $P_n = 12.5(1.05)^{n-1}$

f

$$\begin{aligned} \text{Let } P_n &\geq 2 \times P_1 \\ \therefore \quad 12.5(1.05)^{n-1} &\geq 2 \times 12.5 \\ \therefore \quad (1.05)^{n-1} &\geq 2 \end{aligned}$$

When $n = 15,\ (1.1)^{15-1} = 1.979\,93\ldots < 2$

When $n = 16,\ (1.1)^{16-1} = 2.078\,92\ldots > 2$

It takes 15 years for the population to double.

17 **a** We first calculate $\theta = \angle TSO$. Using the sine rule, we obtain

$$\begin{aligned} \frac{\sin\theta}{6400} &= \frac{\sin 120^\circ}{8000} \\ \sin\theta &= \frac{6400\sin 120^\circ}{8000} \\ &\approx 0.6928 \\ \theta &\approx 43.8538^\circ \end{aligned}$$

Therefore

$$\angle TOS \approx 180 - 120 - 43.8538 = 16.1462^\circ.$$

The satellite completes one orbit every two hours. Therefore, the time in minutes after 12 p.m. will be

$$\frac{16.1462}{360} \times 2 \times 60 = 5.38 \text{ min.}$$

Therefore the time will be approximately 12.05.

b As the satellite rotates, $\angle TOS$ increases. After 6 minutes, the satellite will have rotated by

$$\angle TOS = \frac{6}{120} \times 360^\circ = 18^\circ.$$

We apply the cosine law to find that

$$\begin{aligned} TS &= \sqrt{6400^2 + 8000^2 - 2 \times 6400 \times 8000 \times \cos 18^\circ} \\ &\approx 2752 \text{ km.} \end{aligned}$$

c Let $\angle STO = \theta$. Then using the sine rule, we obtain,

$$\begin{aligned} \frac{\sin\theta}{8000} &= \frac{\sin 18^\circ}{2572} \\ \sin\theta &= \frac{8000\sin 18^\circ}{2572} \\ &\approx 0.8984 \end{aligned}$$

As θ is obtuse, we obtain $\theta \approx 116.0507^\circ$. Therefore, the angle above the horizon will be approximately,

$$116.0507^\circ - 90^\circ \approx 26.1^\circ.$$

18 a Let $\beta = \angle XOB$. Then,

$$\cos\beta = \frac{32}{40}$$
$$\beta = \cos^{-1}\left(\frac{4}{5}\right)$$
$$\approx 0.6435.$$

Therefore, $\angle AOB = 2\beta \approx 1.29$.

b i We use the formula $L = r\theta$

$$\approx 40 \times 1.29$$
$$= 51.48 \text{ cm}.$$

ii We first find the segment area above the surface of the water. This is given by,

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$
$$\approx \frac{1}{2} \times 40^2(1.29 - sin(1.29))$$
$$\approx 261.60 \text{ cm}^2.$$

We subtract this from the area of the full circle to give

$$A \approx \pi \times 40^2 - 261.60 \approx 4764.95 \text{ cm}^2.$$

iii The percentage of the log beneath the surface will be given by

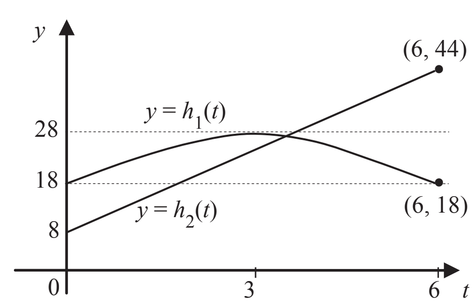
$$\frac{4764.95}{\pi \times 40^2} \times 100\% \approx 94.80\%.$$

19

$$h_1(t) = 18 + 10\sin\left(\frac{\pi t}{6}\right)$$
$$h_2(t) = 8 + 6t$$

a period of $y = h_1(t)$

$$= 2\pi \div \frac{\pi}{6}$$
$$= 12$$



b On a CAS calculator, with $\boldsymbol{f1 = 18 + 10\sin(\pi x/6)}$ and $\boldsymbol{f2 = 8 + 6x}$

The coordinates of the intersection point are (3.311, 27.867). (3.19 am)

c i When $t = 9$ (9.00 am), $h_1(t)$ reaches its minimum value of 8.

ii The original function satisfies this with t redefined, i.e. $h(t) = 8 + 6t$.